

# State augmentation method for non-stationary vibrations of long-span bridges with multi-degree of freedom

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## SUMMARY:

Extreme winds such as hurricanes and thunderstorms often present non-stationary characteristics, having time-varying mean wind speeds and non-stationary wind fluctuations. When concerning wind-induced vibrations under non-stationary wind, the excitation will be a non-stationary process, and the wind-structure coupled system can be represented by a linear time-varying system. The study aims to present a state augmentation method to investigate the non-stationary buffeting of a long-span bridge subjected to non-stationary wind. The unsteady self-excited forces and the non-stationary turbulence-induced forces, i.e., buffeting forces, are included both in this study to predict the flutter and buffeting response. Based on Itô's Formula, the statistical moments of the response are derived through solving a first-order ordinary differential equation system. This study is regarded as subsequent research by the author in order to generalize the previous one in a single degree of freedom case to a multiple degree of freedom case and take the unsteady aerodynamic effects into consideration.

*Keywords: non-stationary wind, buffeting response, Itô's Formula*

## 1. INTRODUCTION

In contrast with the stationary synoptic wind, extreme wind events such as hurricanes and thunderstorms always exhibit considerable non-stationary characteristics. When considering aeroelastic effects, the aerodynamic damping and stiffness will be time-dependent due to the time-varying mean wind speed, and the wind-structure coupled system will be a linear time-varying (LTV) system (L. Hu et al., 2013). In view of these non-stationary effects, many attempts have been made to develop random vibration theory for non-stationary buffeting, including the Monte Carlo method, generalized frequency-domain method, and pseudo excitation method. However, some methods may need intensive calculations due to time-integration process, and the others may be difficult to consider time-dependent system properties.

Based on the theory of Itô's Stochastic Differential Equation, Grigoriu (M. Grigoriu and S. Ariaratnam, 1988) proposed the state augmentation method to calculate the stochastic response

of linear systems subjected to stationary excitations. With this method, the statistical moments of any order of the response can be directly obtained by solving a system of linear differential equations with high efficiency. Although this method has been applied in several wind engineering problems, such as non-Gaussian turbulence (W. Cui et al., 2022), it has been rarely reported for non-stationary buffeting analysis. This paper aims to extend the state augmentation method to investigate the non-stationary buffeting of a long-span bridge subjected to non-stationary wind loads. This study is regarded as subsequent research by the author (Lei et al., 2022) in order to generalize the previous one in a single degree of freedom case to a multiple degree of freedom case and take the unsteady aerodynamic effects into consideration.

## 2. NON-STATIONARY WIND-INDUCED VIBRATIONS OF BRIDGES

### 2.1. Dynamics of Bridge Deck Motions

The motions  $\mathbf{X}_d$  of the full-order bridge deck subjected to self-excited forces  $\mathbf{F}^{se}$  and non-stationary buffeting forces  $\mathbf{F}^b$  are determined by solving the dynamic equation

$$\mathbf{M}_d \ddot{\mathbf{X}}_d + \mathbf{C}_d \dot{\mathbf{X}}_d + \mathbf{K}_d \mathbf{X}_d = \mathbf{F}^{se} + \mathbf{F}^b \quad (1)$$

in which  $\mathbf{M}_d$ ,  $\mathbf{C}_d$  and  $\mathbf{K}_d$  are the mass, damping and stiffness matrices, respectively.

### 2.2. Unsteady Aerodynamic Forces in Time domain

The flutter derivatives are approximated in terms of rational functions known as Roger's approximation. In this case, the unsteady self-excited forces in the time domain are obtained as

$$\mathbf{F}^{se} = \frac{1}{2} \rho U^2(t) \Delta L \left( \mathbf{A}_1^{se} \mathbf{X}_d + \frac{B}{U(t)} \mathbf{A}_2^{se} \dot{\mathbf{X}}_d + \frac{B^2}{U^2(t)} \mathbf{A}_3^{se} \ddot{\mathbf{X}}_d + \sum_{l=1}^{m_{se}} \boldsymbol{\phi}_l^{se} \right) \quad (2a)$$

$$\dot{\boldsymbol{\phi}}_l^{se} = -\frac{d_l^{se} U(t)}{B} \boldsymbol{\phi}_l^{se} + \mathbf{A}_{l+3}^{se} \dot{\mathbf{X}}_d \quad (2b)$$

where  $\mathbf{A}_1^{se}$ ,  $\mathbf{A}_2^{se}$ ,  $\mathbf{A}_3^{se}$ ,  $\mathbf{A}_{l+3}^{se}$  and  $d_l^{se} \geq 0$  are frequency-independent coefficients.  $\boldsymbol{\phi}_l^{se}$  are additional variables introduced to consider the unsteady effect. Note that the time-varying mean wind speed is considered in the aerodynamic coefficients due to its slow variation compared to the primary frequency of the bridge. Analogous to the self-excited forces, the non-stationary buffeting forces in the time domain can be expressed as

$$\mathbf{F}^b = \beta^u(t) \mathbf{F}^{bu} + \beta^w(t) \mathbf{F}^{bw} \quad (3a)$$

$$\mathbf{F}^{bu} = \frac{1}{2} \rho U(t) \Delta L \sum_{l=1}^{m_u+1} \mathbf{A}_l^u \mathbf{u}^s - \frac{1}{2} \rho U(t) \Delta L \sum_{l=1}^{m_u} \frac{d_l^u U(t)}{B} \boldsymbol{\phi}_l^u \quad (3b)$$

$$\dot{\boldsymbol{\phi}}_l^u = -\frac{d_l^u U(t)}{B} \boldsymbol{\phi}_l^u + \mathbf{A}_{l+1}^u \mathbf{u}^s \quad (3c)$$

in which  $\mathbf{A}_1^u$ ,  $\mathbf{A}_{l+1}^u$  and  $d_l^u \geq 0$  are frequency-independent coefficients.  $\boldsymbol{\phi}_l^u$  are additional variables.  $\beta_u(t)$  and  $\beta_w(t)$  are the slowly varying time-modulation functions.  $\mathbf{u}^s$  are the derived stationary Gaussian processes with zero means.

### 2.3. Augmented States of the System and Excitation

With the help of the modal-superposition method, the generalized dynamic equation is derived as

$$\mathbf{M}_q \ddot{\mathbf{q}} + \mathbf{C}_q \dot{\mathbf{q}} + \mathbf{K}_q \mathbf{q} = \mathbf{Q}^{se} + \beta^u \mathbf{Q}^{bu} + \beta^w \mathbf{Q}^{bw} \quad (4)$$

in which  $\mathbf{M}_q$ ,  $\mathbf{C}_q$  and  $\mathbf{K}_q$  are the generalized modal mass, damping and stiffness, respectively.  $\mathbf{q}$  is the modal coordinate vector.  $\mathbf{Q}^{se} = \boldsymbol{\Psi}^T \mathbf{F}^{se}$  is the generalized self-excited force vector.  $\mathbf{Q}^{bu} = \boldsymbol{\Psi}^T \mathbf{F}^{bu}$  and  $\mathbf{Q}^{bw} = \boldsymbol{\Psi}^T \mathbf{F}^{bw}$  are the generalized buffeting force vectors.  $\boldsymbol{\Psi}$  is the matrix of mode shapes. The generalized dynamic equation is transformed to its state-space form

$$\dot{\mathbf{Y}} = \mathcal{A}(t)\mathbf{Y} + \mathbf{B}_u(t)\mathbf{u}^{s,q} + \mathbf{B}_w(t)\mathbf{w}^{s,q} \quad (5)$$

in which  $\mathbf{Y}$  is the state vector.  $\mathcal{A}$ ,  $\mathbf{B}_u$  and  $\mathbf{B}_w$  are matrices of deterministic time-varying coefficients.  $\mathbf{u}^{s,q}$  and  $\mathbf{w}^{s,q}$  are the generalized stationary Gaussian pseudo-fluctuations which can be approximated individually by multivariate Ornstein-Uhlenbeck (OU) processes, i.e.,  $\mathbf{u}^{s,q} \approx \mathbf{Z}^u(t)$  and  $\mathbf{w}^{s,q} \approx \mathbf{Z}^w(t)$ , which satisfy the stochastic differential equation

$$d\mathbf{Z}^u(t) = -\boldsymbol{\alpha}^u \mathbf{Z}^u(t)dt + \boldsymbol{\Theta}^u d\mathbf{W}^u(t) \quad (6a)$$

$$d\mathbf{Z}^w(t) = -\boldsymbol{\alpha}^w \mathbf{Z}^w(t)dt + \boldsymbol{\Theta}^w d\mathbf{W}^w(t) \quad (6b)$$

in which  $\boldsymbol{\alpha}^u$ ,  $\boldsymbol{\Theta}^u$ ,  $\boldsymbol{\alpha}^w$  and  $\boldsymbol{\Theta}^w$  are the corresponding coefficient matrices, and  $\mathbf{W}^u$  and  $\mathbf{W}^w$  are the corresponding Wiener processes which are mutually independent. By substituting Eq. (6) into Eq. (5), the augmented states of the system and the excitations are written as

$$d \begin{bmatrix} \mathbf{Y} \\ \mathbf{Z}^u \\ \mathbf{Z}^w \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathbf{B}_u & \mathbf{B}_w \\ \mathbf{0} & -\boldsymbol{\alpha}^u & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{\alpha}^w \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \mathbf{Z}^u \\ \mathbf{Z}^w \end{bmatrix} dt + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Theta}^u & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Theta}^w \end{bmatrix} d \begin{bmatrix} \mathbf{W}^u \\ \mathbf{W}^w \end{bmatrix} \quad (7)$$

Eq. (7) is usually recognized as an Itô-type stochastic differential equation of the form

$$d\mathbb{Y}(t) = \mathbf{g}(\mathbb{Y}(t), t)dt + \mathbf{h}(\mathbb{Y}(t), t)d\mathbf{W} \quad (8)$$

in which  $\mathbb{Y}$  is the augmented state vector.  $\mathbf{g}$  and  $\mathbf{h}$  are explicit functions of  $\mathbb{Y}$  and time  $t$ .

#### 2.4. Differential Equations Satisfied by Statistical Moments

Assume  $\xi(\mathbb{Y})$  as a scalar-valued function of  $\mathbb{Y}$ , i.e.,  $\xi(\mathbb{Y}) = \prod_i^N \mathbb{Y}_i^{a_i}$  where the superscript  $a_i$  is the non-negative integer exponent. According to Itô's lemma

$$\frac{d}{dt} E[\xi] = E \left[ \frac{\partial \xi}{\partial t} \right] + \sum_i^N E \left[ \frac{\partial \xi}{\partial Y_i} g_i \right] + \frac{1}{2} \sum_i^N \sum_j^N \left[ (\mathbf{h}\mathbf{h}^T)_{ij} \frac{\partial^2 \xi}{\partial Y_j \partial Y_i} \right] \quad (9)$$

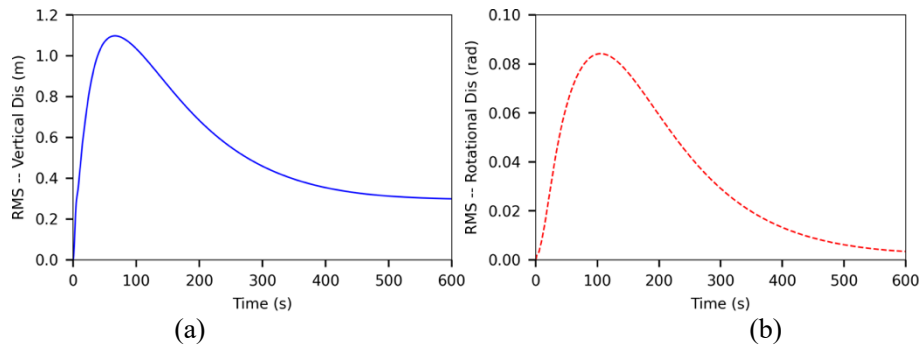
For a prescribed order of  $s = \sum_{i=1}^N a_i$ , substituting all the combinations of  $a_i$  into Eq. (9) gives

$$\dot{\mathbf{m}}_s = \mathbf{P}_1(t)\mathbf{m}_s + \mathbf{Q}_1(t) + \mathbf{P}_2\mathbf{m}_{s-2} + \mathbf{Q}_2 \quad (10)$$

in which  $\mathbf{m}_s = E \left[ \prod_i^N \mathbb{Y}_i^{a_i} \right]$  is the vector of all the statistical moments in the order of  $s$ .  $\mathbf{P}_1$ ,  $\mathbf{P}_2$ ,  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  are the coefficient matrices corresponding to the system properties. The above formulas are the basis of the state augmentation method.

### 3. NUMERICAL APPLICATIONS

As an example of application of the SAM, the proposed method is applied to calculate the non-stationary vibrations of the bridge deck of a long-span bridge (Shenzhen–Zhongshan Bridge in China) subjected to non-stationary wind. The first symmetric torsional mode together with three prior symmetric vertical modes is considered in this study, with the modal frequencies  $n_1 = 0.1049$  Hz ,  $n_2 = 0.1366$ Hz ,  $n_3 = 0.2256$  Hz and  $n_4 = 0.2987$  Hz (torsional frequency), respectively. Simiu and Panofsky type spectra are adopted to represent the along-wind and vertical wind turbulence, with  $u_* = 1.6$  m/s. The time-varying wind speed model is given by  $\bar{U}(t) = (\bar{U}_{\max} - \bar{U}_{\min})(t/t_0) \exp(1 - t/t_0) + \bar{U}_{\min}$  where  $\bar{U}_{\max} = 40$  m/s ,  $\bar{U}_{\min} = 5$  m/s and  $t_0 = 60$  s. Fig. 1 shows the RMS responses for the vertical and rotational displacements at the mid-span. The results are calculated from 0 s to 600 s.



**Figure 1.** Time-varying RMS response at the mid-span: (a) vertical displacement; (b) rotational displacement.

### 4. CONCLUSIONS

This study investigated the non-stationary vibrations of the bridge deck of a long-span bridge subjected to non-stationary unsteady wind loads. Based on Itô's Formula, a state augmentation method is presented to calculate the statistical moments of the non-stationary buffeting response. It is seen that the statistical moments can be directly obtained by solving the first-order ordinary differential equation without going through an intermediate step, e.g., the response spectra.

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